

Temperature Transform of the Boltzmann Equation

R. O. Barrachina,¹ D. H. Fujii,^{1,2} and C. R. Garibotti^{1,3}

Received October 10, 1985; revision received June 11, 1986

We define an integral transform of the energy distribution function for an isotropic and homogeneous diluted gas. It may be interpreted as a linear combination of equilibrium states with variable temperatures. We show that the temporal evolution features of the distribution function are determined by the singularities of this temperature transform. We compare the relaxation to the equilibrium process for Maxwell and very hard-particle interaction models, finding many basic discrepancies. Finally, we formulate an alternative approach, which is given by an N -pole approximation with a clear physical meaning.

KEY WORDS: Boltzmann equation; temperature transform; Maxwell molecules; very hard particles; N -pole approximation.

1. INTRODUCTION

We consider a dilute gas of structureless particles which interact through binary elastic collisions. We want to obtain the corresponding one-particle distribution function $f(\mathbf{r}, \mathbf{p}, t)$, whose temporal evolution is usually characterized by the Boltzmann equation. This is a nonlinear integrodifferential equation with a very complex mathematical structure. In order to gain insight into the relaxation to equilibrium, a great variety of models have been considered in the literature.^(1,2) The simplest real model is obtained by considering spatially homogeneous and isotropic momentum distributions, i.e., $f(\mathbf{r}, \mathbf{p}, t) = f(p, t)$. Under these conditions, significant progress has been achieved in the study of exact solutions of the nonlinear equation. In the recent literature integral transformations have been employed for obtaining more approachable simplifications.

In 1976 Bobylev⁽³⁾ used a Fourier transformation in momentum space

¹ Centro Atómico Bariloche, Comisión Nacional de Energía Atómica, 8400 Bariloche, R.N., Argentina.

² Fellow of the Conselho Nacional de Desenvolvimento Científico e Tecnológico, Brazil.

³ Consejo Nacional de Investigaciones Científicas y Técnicas, Argentina.

and found a particular exact solution (the BKW mode) for Maxwell-type interaction models. However, for non-Maxwellian models the nonlinear collision term is not separable in energy and angle and no major simplification is achieved.⁽⁴⁾ Another transformation technique suitable for Maxwell models was introduced by Alexanian.⁽⁵⁾ This approach offers a deep insight into the characteristics of the temporal behavior of the energy distribution. However, the solution is expressed as a tricky combination of delta functions and their derivatives. A third approach is based on a Laplace transformation in the energy variable. Krook and Wu⁽⁶⁾ applied this method for three-dimensional Maxwell models with isotropic scattering and succeeded in obtaining the BKW solution simultaneously with Bobylev. Afterward this Laplace transform was generalized for a larger special class of Maxwell models, denoted as $g_0(\mu, p)$ models.^(7,8) In spite of not being applicable to all Maxwell models, the Laplace transform method is not strictly limited to them. Actually, by means of a Laplace transformation, Hendriks and Ernst succeeded in solving the Cauchy problem in closed form for a two-dimensional, very hard-particle (VHP) model. However, an alternative approach is required in order to extend the use of this method to other interaction models.

In the present paper we introduce an integral transformation for solving the homogeneous and isotropic nonlinear Boltzmann equation. We show that this transformation has a simple physical meaning and can be directly applied to VHP and Maxwell models. This capability singles out the defined transform from others.

In the following section we state the notation to be used. In Section 3 we introduce our method, which gives a temperature transformed distribution $G(s, t)$ with a quite simple analytical structure. In Section 4 we show that the characteristics and time evolution features of $f(p, t)$ are mainly given by the behavior of the singularities of $G(s, t)$. Using this technique, we study the relaxation toward equilibrium for the very hard-particle model in Section 5, and all Maxwell models in Section 6. The time evolution features for both VHP and Maxwell models are compared. Many basic differences are clearly found by means of the method proposed in this paper.

2. BASIC CONCEPTS AND NOTATION

In 1872 Boltzmann proposed an equation for the temporal evolution of the distribution function $f(\mathbf{r}, \mathbf{p}, t)$ of a diluted gas of particles without internal degrees of freedom,⁽¹⁰⁾

$$\left(\frac{\partial}{\partial t} + \frac{\mathbf{p}}{m} \cdot \nabla_{\mathbf{r}} + \mathbf{F}_e \cdot \nabla_{\mathbf{p}} \right) f(\mathbf{r}, \mathbf{p}, t) = B[f, f] \quad (2.1)$$

For a gas of molecules interacting through binary collisions, the bilinear collision operator reads

$$B[f, f] = \int \frac{g}{m} \sigma(g, \hat{g} \cdot \hat{n}) [f(\mathbf{r}, \mathbf{p}', t) f(\mathbf{r}, \mathbf{p}_1', t) - f(\mathbf{r}, \mathbf{p}, t) f(\mathbf{r}, \mathbf{p}_1, t)] d\mathbf{p}_1 d\hat{n} \quad (2.2)$$

Here $\sigma(g, \hat{g} \cdot \hat{n})$ is the differential cross section for the binary collision $(\mathbf{p}, \mathbf{p}_1) \rightarrow (\mathbf{p}', \mathbf{p}_1')$, with $\mathbf{g} = \mathbf{p}_1 - \mathbf{p}$ and $\mathbf{g}' = \mathbf{p}_1' - \mathbf{p}'$ the relative momenta. From the energy and momentum conservation laws we have⁽²⁾

$$\mathbf{p}' = \frac{1}{2}(\mathbf{p} + \mathbf{p}_1) - \frac{1}{2}|\mathbf{p} - \mathbf{p}_1| \hat{n}, \quad \mathbf{p}_1' = \frac{1}{2}(\mathbf{p} + \mathbf{p}_1) + \frac{1}{2}|\mathbf{p} - \mathbf{p}_1| \hat{n} \quad (2.3)$$

and it follows that $\mathbf{g}' = g \cdot \hat{n}$ (\hat{n} is a unit vector in the direction of \mathbf{g}').

In view of the possible application to systems of one, two, or three dimensions, it is convenient to consider the position \mathbf{r} and momentum \mathbf{p} of the particle of mass m as vectors of arbitrary dimension 2ν .

For a spatially homogeneous distribution function $f(\mathbf{p}, t)$ and without external forces ($\mathbf{F}_e = 0$), the Boltzmann equation reads

$$\frac{\partial}{\partial t} f(\mathbf{p}, t) = B[f, f] \quad (2.4)$$

Conservation of particles, momentum, and energy requires

$$\int f(\mathbf{p}, t) d\mathbf{p} = \eta \quad (2.5)$$

$$\int f(\mathbf{p}, t) \mathbf{p} d\mathbf{p} = 0 \quad (2.6)$$

$$\int f(\mathbf{p}, t) \frac{p^2}{2m} d\mathbf{p} = \nu \eta k T \quad (2.7)$$

where η is the number density and k is Boltzmann's constant. The temperature of the gas is given by the average energy per degree of freedom $\langle \varepsilon \rangle = \nu k T$. The second conservation law indicates that the temporal evolution of the gas is observed from the center-of-mass reference frame $\langle \mathbf{p} \rangle = 0$.

The H -theorem guarantees that $f(\mathbf{p}, t)$ will approach the Maxwell-Boltzmann distribution for any initial condition:

$$f(\mathbf{p}, t) \xrightarrow{t \rightarrow \infty} f_0(\mathbf{p}) = \frac{\eta}{(2\pi m k T)^\nu} e^{-p^2/2mkT} \quad (2.8)$$

The distribution function remains positive for all \mathbf{p} and all $t \geq 0$ when the initial state $f(\mathbf{p}, 0)$ is positive.⁽¹¹⁾

Henceforth, we shall consider isotropic momentum distributions, i.e., $f(p, t)$ with $p = |\mathbf{p}|$. In the energy representation this distribution function is defined in the following way⁽²⁾:

$$F(\varepsilon, t) = \int \delta\left(\varepsilon - \frac{p^2}{2m}\right) f(\mathbf{p}, t) d\mathbf{p} \quad (2.9)$$

That is,

$$F(\varepsilon, t) = \frac{2\pi^v}{\Gamma(v)} m(2m\varepsilon)^{v-1} f[(2m\varepsilon)^{1/2}, t] \quad (2.10)$$

Now the conservation laws are

$$\int_0^\infty F(\varepsilon, t) d\varepsilon = \eta \quad (2.11)$$

$$\int_0^\infty F(\varepsilon, t) \varepsilon d\varepsilon = v\eta kT \quad (2.12)$$

and $F(\varepsilon, t)$ approaches equilibrium as a gamma distribution function:

$$F_0(\varepsilon) = \frac{\eta}{\Gamma(v) kT} \left(\frac{\varepsilon}{kT}\right)^{v-1} \varepsilon^{-\varepsilon/kT} \quad (2.13)$$

In the energy representation Eq. (2.4) may be written in terms of the scattering kernel

$$K(\varepsilon', \varepsilon; E) = \int \frac{d\mathbf{p} d\mathbf{p}_1 d\hat{n} g\sigma(g, \hat{g} \cdot \hat{n})}{m^3 [2\pi^v/\Gamma(v)]^2 [4m^2\varepsilon(E-\varepsilon)]^{v-1}} \\ \times \delta\left(\varepsilon' - \frac{p'^2}{2m}\right) \delta\left(\varepsilon - \frac{p^2}{2m}\right) \delta\left(E - \frac{p^2}{2m} - \frac{p_1^2}{2m}\right) \quad (2.14)$$

which describes the transition probability in the collision $(\varepsilon, E-\varepsilon) \rightarrow (\varepsilon', E-\varepsilon')$. The brackets indicate the energy of the particles before and after the collision. Conservation of the total energy E requires the kernel (2.14) to be null outside the region $\varepsilon, \varepsilon' < E$. Finally, we obtain

$$\frac{\partial}{\partial t} F(\varepsilon, t) = \int_\varepsilon^\infty dE \int_0^E d\varepsilon' [K(\varepsilon, \varepsilon'; E) F(\varepsilon', t) F(E-\varepsilon', t) \\ - K(\varepsilon', \varepsilon; E) F(\varepsilon, t) F(E-\varepsilon, t)] \quad (2.15)$$

3. TEMPERATURE TRANSFORM

We introduce the temperature transform $G(s, t)$ of $F(\varepsilon, t)$ as follows⁽¹²⁾:

$$F(\varepsilon, t) = \frac{1}{2\pi i} \oint_{\Gamma} F_s(\varepsilon) G(s, t) ds \tag{3.1}$$

The path Γ encircles the singularities of $G(s, t)$ in the counterclockwise direction, and $F_s(\varepsilon)$ is the equilibrium gamma distribution function with a modified temperature $(1 - s)T$, i.e.,

$$F_s(\varepsilon) = \frac{\eta}{\Gamma(\nu) kT(1 - s)} \left(\frac{\varepsilon}{kT(1 - s)} \right)^{\nu - 1} e^{-\varepsilon/kT(1 - s)} \tag{3.2}$$

Equation (3.1) is an integral transform as defined by Courant and Hilbert,⁽¹³⁾ where the kernel $F_s(\varepsilon)/F_0(\varepsilon)$ is the generating function of the Laguerre polynomials $L_n^{\nu - 1}(\varepsilon/kT)$. Physically, this transform may be interpreted as a linear superposition of equilibrium states with variable temperatures. $G(s, t)$ represents the weight of each of these states. The conservation laws (2.11) and (2.12) as well as the equilibrium distribution for $G(s, t)$ are easily obtained from Eq. (3.1),

$$\oint_{\Gamma} G(s, t) ds = 2\pi i \tag{3.3}$$

$$\oint_{\Gamma} G(s, t) s ds = 0 \tag{3.4}$$

$$G(s, t) \xrightarrow{t \rightarrow \infty} G_0(s) = 1/s \tag{3.5}$$

Now we will discuss the relation of $G(s, t)$ to other integral transforms considered in the recent literature.⁽²⁻⁹⁾ We will show that $G(s, t)$ results from successive Laplace and Fourier transformations of $F(\varepsilon, t)$. The Fourier transform introduced by Bobylev⁽³⁾ has proved to be the most general and useful one, leading to a simple treatment of the nonlinear Boltzmann equation for Maxwell interaction models. In the momentum representation this transform reads

$$\phi(\mathbf{r}, \mathbf{q}, t) = \int f(\mathbf{r}, \mathbf{p}, t) \exp(-i\mathbf{p} \cdot \mathbf{q}) d\mathbf{p} \tag{3.6}$$

and

$$f(\mathbf{r}, \mathbf{p}, t) = \frac{1}{(2\pi)^{2\nu}} \int \phi(\mathbf{r}, \mathbf{q}, t) \exp(i\mathbf{p} \cdot \mathbf{q}) d\mathbf{q} \tag{3.7}$$

The Fourier transform of a spatially uniform system with an isotropic momentum distribution may be written in terms of $F(\varepsilon, t)$. Defining $\phi(q, t) = \eta\psi(\frac{1}{2}mkTq^2, t)$, we have

$$\psi(z, t) = \frac{1}{\eta} \int_0^\infty {}_0F_1\left(v, -\frac{z\varepsilon}{kT}\right) F(\varepsilon, t) d\varepsilon \quad (3.8)$$

and

$$F(\varepsilon, t) = \frac{\eta}{\Gamma(v)^2 kT} \int_0^\infty \left(\frac{z\varepsilon}{kT}\right)^{v-1} {}_0F_1\left(v, -\frac{z\varepsilon}{kT}\right) \psi(z, t) dz \quad (3.9)$$

The distribution at equilibrium is $\psi_0(z)$, where we define $\psi_s(z)$ with a modified temperature $(1-s)T$:

$$\psi_s(z) = e^{-(1-s)z} \quad (3.10)$$

A sufficient condition for the existence of $\psi(z, t)$ is that the distribution function belongs to the Hilbert space $\mathcal{L}_2(0, \infty)$: the space of square-integrable functions defined on the interval $0 \leq \varepsilon \leq \infty$, with norm⁽²⁾

$$\|F\|^2 = \int_0^\infty |F(\varepsilon, t)|^2 \frac{d\varepsilon}{F_0(\varepsilon)} < \infty \quad (3.11)$$

There is no physical reason to prevent consideration of a distribution function not belonging to this Hilbert space.⁽³⁾ However, we shall restrict our attention to this space, unless otherwise stated.

In order to relate the characteristic function $\psi(z, t)$ with our transform $G(s, t)$ [Eq. (3.1)], we require $\psi(z, t)/\psi_0(z)$ to belong to Ω_γ (as defined in the Appendix). Then its Laplace transform

$$G(s, t) = \int_0^\infty e^{-z(s-1)} \psi(z, t) dz \quad (3.12)$$

belongs to ω_γ , and satisfies the following Pincherle inversion formula:

$$\psi(z, t) = \frac{1}{2\pi i} \oint_{\Gamma} \psi_s(z) G(s, t) ds \quad (3.13)$$

Replacing this expression in Eq. (3.9), it is easy to show that $G(s, t)$ is just our temperature transform.

The representation of $G(s, t)$ given by Eq. (3.12) is similar to the generalized Laplace transform introduced by Ernst for $g_0(\mu, p)$ models.⁽⁷⁾ For $p=1$ both integral transforms are the same, except for a change of coordinates.

Since $\psi(z, t)/\psi_0(z)$ is required to be an entire function, its MacLaurin series

$$\psi(z, t) = \psi_0(z) \sum_{n=0}^{\infty} \frac{c_n(t)}{n!} z^n \tag{3.14}$$

converges and represents $\psi(z, t)$ for all complex z . Then from Eqs. (3.12) and (3.14) we have

$$G(s, t) = \sum_{n=0}^{\infty} \frac{c_n(t)}{s^{n+1}} \tag{3.15}$$

This expansion is analytic outside a circle $|s| = \gamma$, but not outside any smaller circle (i.e., belongs to ω_γ), with

$$\gamma(t) = \limsup_{n \rightarrow \infty} |c_n(t)|^{1/n} \tag{3.16}$$

We also point out the relation with another integral transform, i.e., Alexanian's transform $A(\theta, t)$, which reads⁽⁵⁾

$$F(\varepsilon, t) = \int_{-\infty}^1 F_\theta(\varepsilon) A(\theta, t) d\theta \tag{3.17}$$

with the following conservation rules and asymptotic condition:

$$\int_{-\infty}^1 A(\theta, t) d\theta = 1 \tag{3.18}$$

$$\int_{-\infty}^1 A(\theta, t) \theta d\theta = 0 \tag{3.19}$$

$$A(\theta, \infty) = \delta(\theta) \tag{3.20}$$

It should be noted that this transform may be considered as another temperature distribution function. However, delta-type singularities make $A(\theta, t)$ a tricky mathematical tool.⁽²⁾ The relation between $G(s, t)$ and $A(\theta, t)$ can be obtained from Eqs. (3.8), (3.12), and (3.17):

$$G(s, t) = \int_{-\infty}^1 \frac{A(\theta, t)}{s - \theta} d\theta \tag{3.21}$$

This shows that delta-type singularities of $A(\theta, t)$ are replaced by complex poles in $G(s, t)$, allowing the use of powerful mathematical methods such as Heaviside's expansion theorem⁽¹⁴⁾ and the technique of rational approximants.⁽¹⁵⁾

4. SINGULARITIES OF THE TEMPERATURE TRANSFORM

Equation (3.1) shows that the time evolution features of $F(\varepsilon, t)$ are mainly given by the behavior of the singularities of the temperature transform function. In this section we shall discuss some relevant characteristics of these singularities.

Here, we consider that the only singularities of $G(s, t)$ are poles on the complex plane. That means that the distribution has an exponential decay at large energy. From the physical point of view a potential decay is allowed, but in this case $F(\varepsilon, t)$ will not belong to the $\mathcal{L}_2(0, \infty)$ space defined by Eq. (3.11), and $G(s, t)$ will have cuts in the complex s -plane. Actually, a cut line might go beyond the nonanalyticity circle $|s| = \gamma$. This is an interesting point, which will not be discussed in this paper.

The discussion in Section 3 suggests that the localization of the polar singularities of the temperature transform function would be related to the asymptotic behavior of the characteristic function $\psi(z, t)$. Let us introduce the concept of indicator function $\gamma(\phi, t)$ of $\psi(z, t)$ ⁽¹⁴⁾: for a fixed time t , $\gamma(\phi, t)$ denotes the infimum of all real numbers $\alpha(t)$ such that

$$\left| \frac{\psi(\rho e^{i\phi}, t)}{\psi_0(\rho e^{i\phi})} \right| < e^{\alpha\rho} \quad (4.1)$$

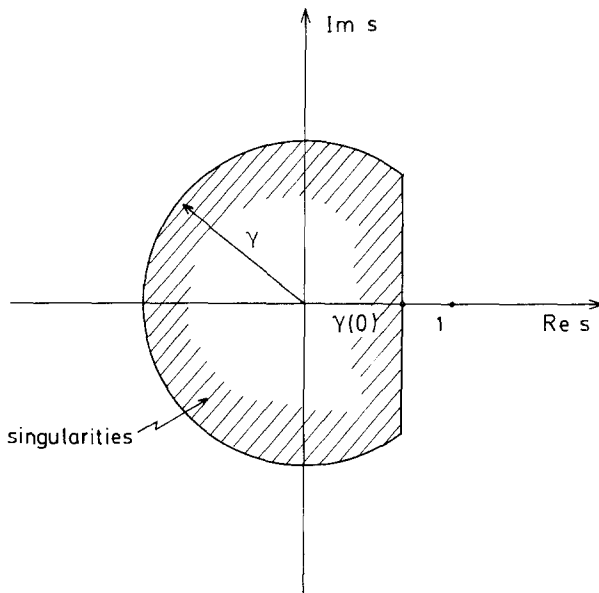


Fig. 1. Location of the singularities of the temperature transform $G(s, t)$.

for all sufficiently large $\rho \geq 0$. Hence, $\gamma(\phi, t) \leq \gamma(t)$. Actually, $G(s, t)$ is analytic outside the closed half-plane $\text{Re}(se^{i\phi}) \leq \gamma(\phi, t)$ for each ϕ . The intersection of all these half-planes containing all singular points of $G(s, t)$ defines the convex hull of the singularities of $G(s, t)$. Since $F(\varepsilon, t)$ is real, $\psi^*(z^*, t) = \psi(z, t)$, and $\gamma(\phi, t)$ is an even function of ϕ ; i.e., the convex hull is symmetric with respect to the real s axis. In conclusion, the singular points of $G(s, t)$ lie on the real axis or join in complex conjugate pairs. Furthermore, the requirement of $F(\varepsilon, t)$ belonging to $\mathcal{L}_2(0, \infty)$ restricts the convex hull to the region $|\gamma(0, t)| < 1$ (Fig. 1).

We have shown that the time evolution of the distribution function $F(\varepsilon, t)$ is mainly determined by the behavior of the singular points of the temperature transform $G(s, t)$. For the case of N simple poles, this function may be written

$$G(s, t) = \sum_{n=1}^N \frac{\alpha_n(t)}{s - \sigma_n(t)} \tag{4.2}$$

where the conservation laws (3.3) and (3.4) impose

$$\sum_{n=1}^N \alpha_n = 1 \tag{4.3}$$

$$\sum_{n=1}^N \alpha_n \sigma_n = 0 \tag{4.4}$$

Applying Eq. (3.1) to Eq. (4.2), we can write $F(\varepsilon, t)$ in terms of the singularities of $G(s, t)$:

$$F(\varepsilon, t) = \sum_{n=1}^N \alpha_n(t) F_{\sigma_n(t)}(\varepsilon) \tag{4.5}$$

This formula expresses $F(\varepsilon, t)$ as an expansion in equilibrium gamma distributions (3.2) with modified temperatures given by the location of the poles of the corresponding transform $G(s, t)$, $T_n = (1 - \sigma_n)T$, and modified number densities fixed by the respective residues $\eta_n = \alpha_n \eta$. Therefore, the analysis of the singularities of the temperature transform function describe the evolution of the distribution function.

Up to now we have been concerned with the location and some characteristics of these singularities. In the rest of this paper we shall aim at a study of their temporal evolution for different interaction models, i.e., the very hard-particle model in Section 5 and the Maxwell models in Section 6.

5. VERY HARD-PARTICLE MODEL

We consider an interaction model which allows a closed analytical solution of the corresponding nonlinear Boltzmann equation for initial conditions with arbitrary exponential decay. These analytical solution will give an explicit confirmation of many assumptions sketched in the previous section as well as show new properties of the temperature transform $G(s, t)$.

We analyze a two-dimensional system of very hard particles characterized by a collision cross section

$$\sigma(g, \hat{g} \cdot \hat{n}) = \frac{\mu}{4} \frac{g}{\eta k T} [1 - (\hat{g} \cdot \hat{n})^2]^{1/2} \quad (5.1)$$

This interaction model was introduced by Hendriks and Ernst.⁽⁹⁾ Mathematically most of this section follows their work.

The corresponding Boltzmann equation reads

$$\left(\frac{1}{\mu} \frac{\partial}{\partial t} + \frac{\varepsilon}{kT} + 1 \right) F(\varepsilon, t) = \frac{1}{\eta k T} \int_{\varepsilon}^{\infty} dE \int_0^E d\varepsilon' F(\varepsilon', t) F(E - \varepsilon', t) \quad (5.2)$$

Through a change of variables the temperature transformation of Eq. (5.2) gives an ordinary differential equation of Riccati type with solution

$$G(s, t) = \frac{1}{s} \left[1 + \frac{e^{-\mu t}}{(s-1)[s\varphi(r) - s(s-1)e^{-\mu t}]} \right] \quad (5.3)$$

where

$$r(s, t) = \frac{s + \mu t(s-1)}{1 + \mu t(s-1)} \quad (5.4)$$

and

$$\varphi(r) = \frac{1}{r} \left[(r-1) + \frac{1}{(r-1)[rG(r, 0) - 1]} \right] \quad (5.5)$$

This solution will allow us to find for this particular model the characteristic features of the time evolution of the singular points of $G(s, t)$. Equation (5.3) shows its singularities moving toward the point $s=1$ and fading out as time elapses. This characteristic point corresponds to a null temperature and henceforth will be called the escape point. However, there is one first-order pole corresponding to a solution of the equation

$$s\varphi(r) - (s-1)e^{-\mu t} = 0 \quad (5.6)$$

which approaches the equilibrium point $s = 0$ for large t . This pole is commonly described as the Maxwell pole of $G(s, t)$, since it gradually builds up the Maxwell equilibrium distribution function $F_0(\varepsilon)$. At large values of the energy the approach of $F(\varepsilon, t)$ to $F_0(\varepsilon)$ will be from above or below, depending on whether the equilibrium point $s = 0$ is reached by the Maxwell pole from the left or the right, respectively.

Now we shall consider some particular initial distributions in order to gain further insight into the relaxation process toward equilibrium. One of the simplest examples of initial distributions is a superposition of N Maxwellians with different temperatures

$$F(\varepsilon, 0) = \sum_{n=1}^N \alpha_n(0) F_{\sigma_n(0)}(\varepsilon) \tag{5.7}$$

This is equivalent to the presence of N simple poles in the corresponding temperature transform

$$G(s, 0) = \sum_{n=1}^N \frac{\alpha_n(0)}{s - \sigma_n(0)} \tag{5.8}$$

A notable result is that this distribution keeps its structure at all times. It remains as a linear combination of N Maxwellians. The real pole farther left is the Maxwell pole and moves toward the equilibrium point $s = 0$, while the rest approach the escape point $s = 1$.

An interesting example of such initial conditions is given by the combination of a pair of complex poles with a real one in the corresponding temperature transform. The presence of such a pair of complex poles corresponds to an oscillatory behavior of the distribution function. This is the case of the function

$$F(\varepsilon, 0) = \frac{\beta\eta}{kT} e^{-a(\varepsilon/kT)} \left[1 - \cos\left(b \frac{\varepsilon}{kT}\right) \right] \tag{5.9}$$

Normalization of particle number and total energy requires the following relations between the parameters:

$$b = a \left(\frac{3-a}{a-1} \right)^{1/2} \tag{5.10}$$

$$\beta = \frac{2a}{3-a} \tag{5.11}$$

The corresponding temperature transform has three poles σ_1 , σ_2 , and σ_3 , which initially are given by

$$\sigma_1(0) = \frac{a-1}{a} \tag{5.12}$$

$$\sigma_2(0) = \sigma_3^*(0) = 1 - \frac{1}{a-ib} \tag{5.13}$$

For the case $a = 1.4$ we have plotted the stroboscopic images of the poles and the corresponding residues in Fig. 2. We observe the motion described above. The real pole σ_1 moves toward the equilibrium point, while the

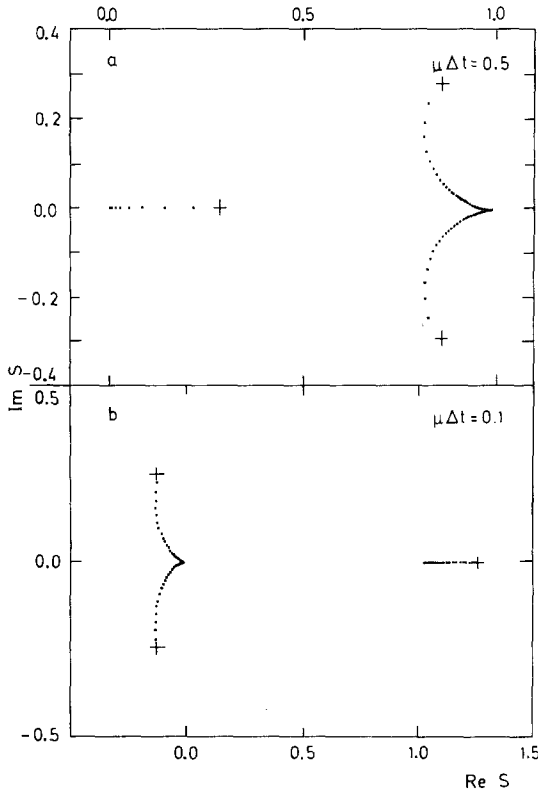


Fig. 2. Stroboscopic temporal images in the complex s plane of (a) poles and (b) residues for the VHP model with the initial condition given by Eqs. (5.9)–(5.11), with $a = 1.4$. The initial position is indicated by a plus sign.

complex poles go to the escape point. The time evolution of the deviation from equilibrium

$$R(\epsilon, t) = \frac{F(\epsilon, t) - F_0(\epsilon)}{F_0(\epsilon)} \tag{5.14}$$

is shown in Fig. 3.

An approximate solution of the nonlinear equation is usually desired. For small deviations from equilibrium, we can neglect quadratic terms in Eq. (5.2), obtaining the following linearized Boltzmann equation:

$$\left(\frac{1}{\mu} + \frac{\partial}{\partial t} + \frac{\epsilon}{kT} + 1 \right) R(\epsilon, t) = \int_0^\epsilon \frac{F_0(\epsilon) - F_0(\epsilon')}{F_0(\epsilon)} R(\epsilon', t) d\epsilon' \tag{5.15}$$

This equation can be solved by the same method previously applied to the nonlinear equation. The solution is

$$G_L(s, t) = \frac{1}{s} \left[1 + \frac{e^{-\mu t}}{s(s-1)} B(r) \right] \tag{5.16}$$

where $r(s, t)$ is given by Eq. (5.4), and the arbitrary function $B(r)$ is determined by the initial distribution $G(s, 0)$:

$$B(r) = r(r-1)[rG(r, 0) - 1] \tag{5.17}$$

with r given by Eq. (5.4). Introducing Eqs. (5.8) and (5.17) into Eq. (5.16), it is easy to prove that for large times all the N poles of the initial dis-

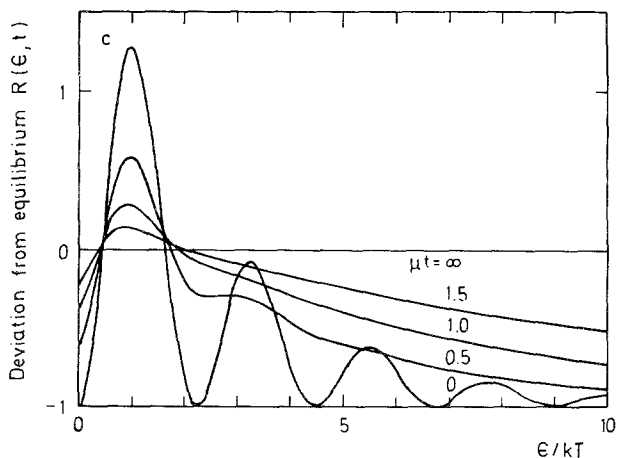


Fig. 3. Deviation from equilibrium for the case described in Fig. 2.

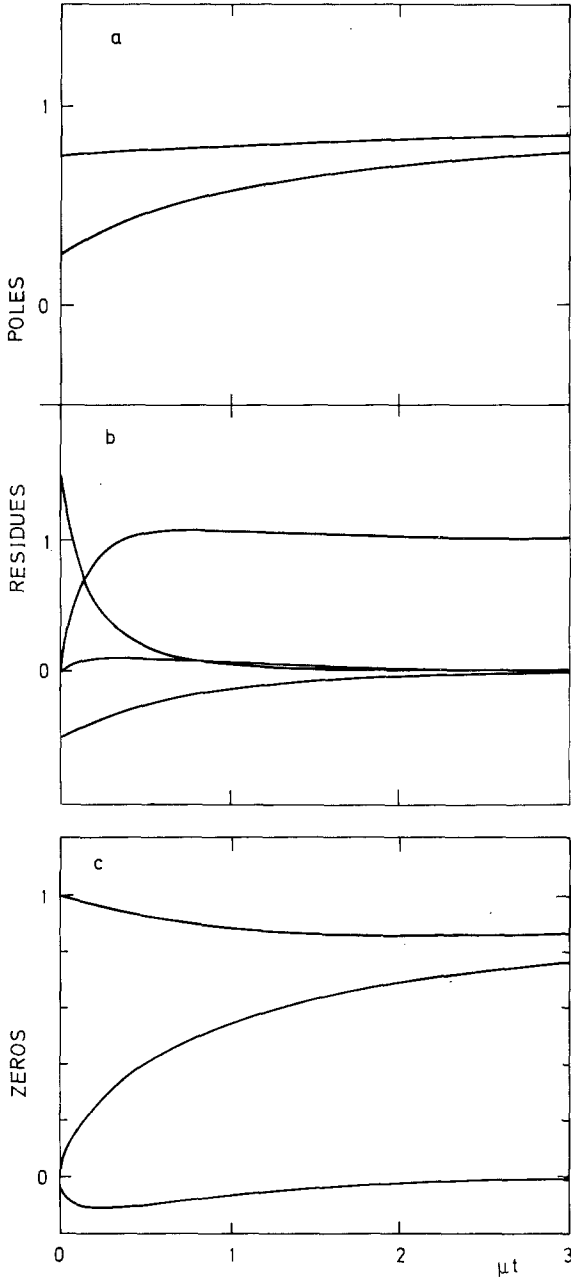


Fig. 4. Time evolution of the VHP linearized solution with initial conditions given by Eq. (5.7) and $\sigma_1(0) = 1/4$, $\sigma_2(0) = 3/4$. (a) Poles, (b) residues, and (c) zeros of $G(s, t)$.

tribution (5.8) move toward the escape point $s = 1$ for large time, according to

$$\sigma_n(t) = \frac{\sigma_n(0) - [\sigma_n(0) - 1] \mu t}{1 - [\sigma_n(0) - 1] \mu t} \tag{5.18}$$

However, Eq. (5.16) shows no Maxwell pole approaching the equilibrium point $s = 0$. Now, a double pole appears at this point and rests there for $t = 0$. The time evolution of the temperature transform for the linearized case is given by

$$G_L(s, t) = \frac{\beta_1(t)}{s} + \frac{\beta_2(t)}{s^2} + \sum_{n=1}^N \frac{\alpha_n(t)}{s - \sigma_n(t)} \tag{5.19}$$

where all the residues approach zero asymptotically, except for $\beta_1(t)$, which approaches 1. It is obvious that initially $\beta_1(0) = \beta_2(0) = 0$. The appearance of a double pole in the equilibrium point has to be accompanied by the appearance of a double zero at that point. Afterward, it branches out into two simple zeros. One of them, after moving away, comes back to the equilibrium point asymptotically, while the other zero moves toward the escape point $s = 1$. These characteristics of the time evolution of the linearized solution for the very hard particle model are shown in Figs. 4 and 5.

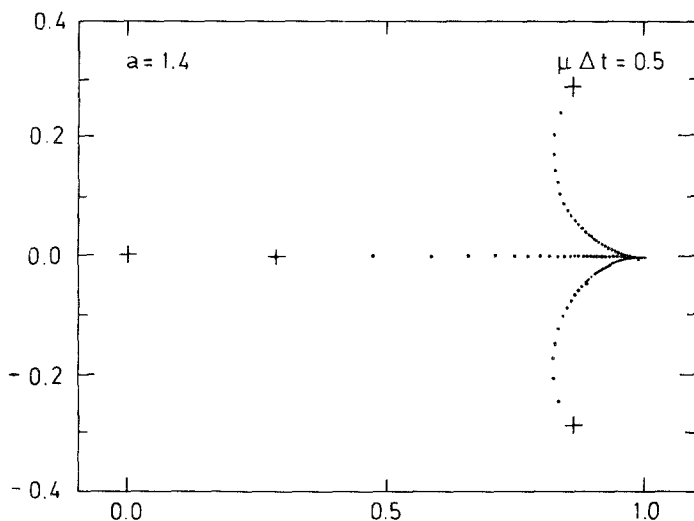


Fig. 5. As in Fig. 2 when the linearized approximation is used.

6. MAXWELL MODELS

In order to simplify the complex mathematical structure of the collision term in Eq. (2.2), we look for mathematical models of the cross section with a simple dependence on the relative momentum and the scattering angle. Maxwell models provide a large simplification of the Boltzmann equation, since the collision probability depends on the scattering angle and not on the relative momentum:

$$\frac{g}{m} \sigma(g, \hat{g}, \hat{n}) = \alpha(\hat{g} \cdot \hat{n}) \tag{6.1}$$

A scattering cross section with this characteristic may be obtained from a repulsive interaction potential $V(r) = ar^{-s}$ with $s = 2(2v - 1)$. The probability $\alpha(\hat{g} \cdot \hat{n})$ becomes a rather complicated function of its arguments.⁽¹⁰⁾ However, we may define more general mathematical models by a convenient choice of the function $\alpha(\hat{g} \cdot \hat{n})$. In general, the models so introduced are not derived from a two-body interaction potential. A quite interesting class of such models is given by the choice⁽²⁾

$$\alpha(\cos \theta) = \mu \Gamma(v - \frac{1}{2}) (\frac{1}{2} \sin \theta)^{2(p-v)+1} / \eta (4\pi)^{v-1/2} B(p, p)$$

[$g_0(\mu, p)$ models] for which the temperature transformed Boltzmann equation can be easily obtained:

$$\left(\frac{1}{\mu} \frac{\partial}{\partial t} + 1 \right) \frac{\partial^{2p-1}}{\partial s^{2p-1}} G(s, t) = - \frac{1}{B(p, p)} \left[\frac{\partial^{p-1}}{\partial s^{p-1}} G(s, t) \right]^2 \tag{6.2}$$

A similar simple equation can be proved for the delta [$\alpha(\cos \theta) \sim \delta(\cos \theta)$] and other Maxwell models. However, we shall follow a more general and useful approach: The solution in the $\mathcal{L}_2(0, \infty)$ space of the nonlinear Boltzmann equation for isotropic Maxwell models can be written as an expansion in generalized Laguerre polynomials⁽²⁾:

$$F(\varepsilon, t) = F_0(\varepsilon) \sum_{n=0}^{\infty} C_n(t) L_n^{v-1}(\varepsilon/kT) \tag{6.3}$$

which is equivalent to a Laurent expansion of $G(s, t)$, Eq. (3.15). Putting this into the Boltzmann equation, we obtain the following infinite set of equations for the moments $C_n(t)$,⁽²⁾

$$C_0(t) = 1 \tag{6.4}$$

$$C_1(t) = 0 \tag{6.5}$$

$$C_n(t) = e^{-\Lambda_n t} \left[C_n(0) + \int_0^t e^{\Lambda_n \tau} \sum_{m=1}^{n-1} \mu_{nm} C_m(\tau) C_{n-m}(\tau) d\tau \right] \tag{6.6}$$

where the real coefficients μ_{nm} are given in terms of the scattering cross section (6.1) by

$$\mu_{nm} = \eta \binom{n}{m} \int \alpha(\cos \theta) \left(\cos \frac{\theta}{2}\right)^{2(n-m)} \left(\sin \frac{\theta}{2}\right)^{2m} d\hat{n} \quad (6.7)$$

and A_n are the eigenvalues of the linearized collision operator

$$A_n = \eta \int \alpha(\cos \theta) \left[1 + \delta_{n0} - \left(\cos \frac{\theta}{2}\right)^{2n} - \left(\sin \frac{\theta}{2}\right)^{2n} \right] d\hat{n} \quad (6.8)$$

Expression (6.6) gives the general solution of the nonlinear Boltzmann equation for Maxwell models, with initial conditions in the Hilbert space $\mathcal{L}_2(0, \infty)$. Several authors have investigated these solutions.^(2,4,15) They found sufficient conditions for the existence of such solutions and the absolute convergence of the expansion given by Eq. (6.3).^(16,17) Iterative solution of the moment equation has been considered in a previous paper.⁽¹⁸⁾

Applying the Cauchy-Hadamard formula (3.16) to expression (6.6), it is possible to show that for an initial condition (5.7) the growth parameter $\gamma(t)$ is time independent. All the singularities of $G(s, t)$ are restricted to $|s| \leq \gamma$ with $\gamma = \max_n |\sigma_n(0)|$. It indicates a completely different behavior with respect to the time evolution displayed by the very hard-particle model, where $\gamma(t) \rightarrow_{t \rightarrow \infty} 1$.

A different type of initial distribution is given by the BKW mode. This mode has been obtained by Bobylev⁽³⁾ and Krook and Wu⁽⁶⁾ as a particular exact solution for a velocity isotropic and spatially uniform gas with Maxwell-type interactions. It is given by

$$F(\varepsilon, t) = F_\sigma(\varepsilon) \left[1 - \frac{\sigma}{1 - \sigma} \left(v - \frac{\varepsilon}{kT(1 - \sigma)} \right) \right] \quad (6.9)$$

with

$$\sigma(t) = \sigma(0) e^{-A_2 t/2} \quad (6.10)$$

We see that the BKW mode constitutes a class of particular solutions, which is parametrized by the constant $\sigma(0)$. This coefficient has the following constraint imposed by the positivity of the solution:

$$0 < \sigma(0) < 1/(1 + \nu) \quad (6.11)$$

The particular Maxwell interaction model is fixed by the eigenvalue. The moments of the Laguerre series (6.2) are given by

$$C_n(t) = (1 - n)[\sigma(t)]^n \quad (6.12)$$

Then from Eq. (3.16) we obtain the temperature transform

$$G(s, t) = (s - 2\sigma)/(s - \sigma)^2 \quad (6.13)$$

which has a real double pole decaying exponentially toward the equilibrium point according to Eq. (6.10). For this particular distribution function the number of poles of the temperature transform is conserved in time. However, and except for certain particular interaction models and initial conditions, new poles will appear as time elapses.

Let us consider that the initial temperature transform has N simple poles. For Maxwell models the number of poles will not be conserved in time. However, we shall suppose that even in these models the distribution function can be adequately approximated by a combination of N simple poles at any time:

$$G_N(s, t) = \sum_{m=1}^N \frac{\alpha_m(t)}{s - \sigma_m(t)} \quad (6.14)$$

This is equal to Eq. (4.2), but now it would be considered as an approximation to the true $G(s, t)$, namely the Padé approximant $[N-1/N]$.⁽¹⁹⁾ We will define the poles and residues by requiring the rational approximant to be equal to the temperature transform expansion up to the $2N$ th order in $1/s$. Comparing Eq. (6.14) with the Laurent expansion (3.15) and equating the coefficients of the same powers of $1/s$, we obtain the following system of coupled equations:

$$\sum_{m=1}^N \alpha_m (\sigma_m)^n = C_n(t); \quad 0 \leq n < 2N \quad (6.15)$$

In principle this system would allow us to calculate the poles σ_m and residues α_m of the rational approximant (6.14), starting from the moments (6.6). It is important to note that the usual method for solving the system (6.15) (namely the product-difference algorithm of Gordon⁽²⁰⁾) cannot be applied here. In view of the conservation laws of particle number and total energy, the first two moments are $C_0(t) = 1$ and $C_1(t) = 0$, and therefore the algorithm fails. However, it is easy to show that the poles σ_m are roots of the characteristic equation⁽¹⁹⁾

$$P(x) = \det \begin{bmatrix} C_0 & C_1 & \cdots & C_N \\ C_1 & C_2 & & \vdots \\ \vdots & & & \\ C_{N-1} & & & C_{2N-1} \\ 1 & x & \cdots & x^N \end{bmatrix} \quad (6.16)$$

When these poles have been obtained, the respective residues may be calculated as the solution of the following Vandermonde linear system:

$$\begin{bmatrix} 1 & \cdots & 1 \\ \sigma_1 & \cdots & \sigma_N \\ \vdots & & \vdots \\ \sigma_1^{N-1} & \cdots & \sigma_N^{N-1} \end{bmatrix} \cdot \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_N \end{bmatrix} = \begin{bmatrix} C_0 \\ C_1 \\ \vdots \\ C_N \end{bmatrix} \quad (6.17)$$

This straightforward method is adequate for low values of N , say $N \leq 10$. For higher values of N a more refined technique is required, i.e., some modified version of the quotient-difference algorithm of Rutishauser.⁽²¹⁾

The use of the Padé approximant (6.14) is equivalent to approximating the distribution function $F(\epsilon, t)$ with a linear combination of equilibrium gamma distributions (3.2) with time-dependent temperatures and densities. This gives for $F(\epsilon, t)$ an expression equal to Eq. (4.5). As described in Section 4, the expansion (4.5) has a clear physical interpretation. We can ask: "Why do we use this approximant instead of the Laguerre series (6.3)?" In fact, for a given initial condition, the solution of the Boltzmann equation can be found by the analytical resolution of the system (6.6) for the moment $C_n(t)$ up to a certain order $n \leq N$. However, from the asymptotic behavior

$$L_n^{v-1} \left(\frac{\epsilon}{kT} \right) \sim \frac{1}{n!} \left(-\frac{\epsilon}{kT} \right)^n; \quad \epsilon \gg kT \quad (6.18)$$

it is clear that the truncated Laguerre series calculation is necessarily restricted to low values of energy $\epsilon < E(N, t)$. This E is an increasing function of the truncation order N and time t . Therefore, it is desirable to improve the numerical convergence, especially for small times. With respect to this point, Barnsley and Turchetti⁽²²⁾ have shown the convenience of using generalized Padé approximants. We are not particularly interested in the convergence of these approximants, but in the temporal evolution of the poles and residues of the corresponding temperature transform given by Eq. (6.14). Since $C_n/C_m \rightarrow_{t \rightarrow \infty} 0$ for $n > m \neq 1$, it is quite easy to show that all the roots of the characteristic equation (6.16) tend to zero as time elapses. This is a notable result, since it shows a quite different behavior than the very hard-particle model. In that case all the poles move toward the escape point except for one pole (the Maxwell pole), which moves toward the equilibrium point. Here all the singularities behave as Maxwell poles exponentially approaching the equilibrium point.

The simplest example is given by the bipolar approximant $G_2(s, t)$. The characteristic equation reads

$$P(x) = \det \begin{bmatrix} 1 & 0 & C_2 \\ 0 & C_2 & C_3 \\ 1 & x & x^2 \end{bmatrix} = C_2 x^2 - C_3 x - C_2^2 = 0 \quad (6.19)$$

Furthermore, we know that the eigenvalue spectrum shows only one degeneracy,⁽²⁾ i.e.,

$$3A_2 = 2A_3 \quad (6.20)$$

Then the change of variables $y = xe^{A_2 t/2}$ in Eq. (6.19) gives

$$C_2(0)y^2 - C_3(0)y - C_2(0)^2 = 0 \quad (6.21)$$

We conclude that the time dependence of the poles is simply given by the exponential factor $e^{-A_2 t/2}$, namely

$$\sigma_1(t) = \sigma_1(0) e^{-A_2 t/2}; \quad \sigma_2(t) = \sigma_2(0) e^{-A_2 t/2} \quad (6.22)$$

The residues are obtained from Eq. (6.17). They are time independent:

$$\alpha_1(t) = \frac{\sigma_2(0)}{\sigma_2(0) - \sigma_1(0)}; \quad \alpha_2(t) = \frac{\sigma_1(0)}{\sigma_1(0) - \sigma_2(0)} \quad (6.23)$$

7. CONCLUSIONS

In the present paper we have introduced an integral transform of the distribution function for the velocity isotropic and spatially homogeneous case. We show the relation of this method to the following transformations: (1) the Fourier transform employed by Bobylev, (2) the Laplace transform applied by Krook and Wu, and (3) the transform formulated by Alexanian. These latter transformations are devised for particular interaction models; we show that the temperature transform has a straightforward application to the models considered here, namely the Maxwell and very hard-particle models. The approach shows that the characteristics and time evolution features of the velocity or energy distribution of particles are mainly determined by the behavior of the singularities of the temperature transform. The relaxation to equilibrium of the singular points can be obtained under very general conditions and depends on the specific interaction model and not on the initial conditions imposed on the distribution. The temperature transform does not lead to any new relaxation phenomena in the con-

sidered cases, but the pole movements give a quite elegant and unified vision of the approach to equilibrium. We find that for the very hard-particle model there is a single pole which determines the relaxation process, and for the Maxwell interaction case all the poles contribute to the large-time behavior. We remark that the trajectories of the poles determine the evolution of the system. Our discussion is confined to single poles even when more complex singularities could be analyzed.

APPENDIX

Here we will define the space Ω_γ of entire functions of exponential type with growth parameter γ . We say that $f(z)$ belongs to Ω_γ when:

1. $f(z)$ is an entire function in the z -complex plane,

$$f(z) = \sum_{n=0}^{\infty} \frac{C_n}{n!} z^n \tag{A1}$$

2. $f(z)$ is of exponential type, which means that there exists $\alpha > 0$ such that

$$\max_{|z|=\rho} |f(z)| < e^{\alpha\rho} \tag{A2}$$

for all sufficiently large ρ . The infimum γ of all these numbers α will be called the growth parameter of $f(z)$.

When $f(z) \in \Omega_\gamma$ it has a Laplace transform⁽¹⁴⁾

$$G(s) = \int_0^\infty e^{-zs} f(z) dz \tag{A3}$$

well defined and analytic, at least for $\text{Re}(s) > \gamma$. Let ω_γ be the class of functions of a complex variable s , analytic outside the circle $|s| = \gamma$ and outside no smaller circle, and vanishing at infinity. It may be shown that for every $\gamma \geq 0$, Eq. (A3) defines a one-to-one mapping of Ω_γ onto ω_γ .⁽¹⁴⁾ Pincherle's theorem gives an inversion formula for Eq. (A3):

$$f(z) = \frac{1}{2\pi i} \oint_\Gamma e^{zs} G(s) ds \tag{A4}$$

where Γ encircles the set $|s| \leq \gamma$ in the counterclockwise sense. From Eqs. (A3) and (A1) we have (Hardy's theorem)

$$G(s) = \sum_{n=0}^{\infty} \frac{C_n}{s^{n+1}} \tag{A5}$$

REFERENCES

1. J. L. Lebowitz and E. W. Montroll, eds., *Non-equilibrium Phenomena I, The Boltzmann Equation* (North-Holland, Amsterdam, 1983).
2. M. H. Ernst, *Phys. Rep.* **78**:1 (1981).
3. A. V. Bobylev, *Sov. Phys. Dokl.* **20**:820, 822 (1976).
4. E. H. Hauge and E. Praestgaard, *J. Stat. Phys.* **24**:21 (1981).
5. M. Alexanian, *Phys. Lett.* **74A**:1 (1979).
6. M. Krook and T. T. Wu, *Phys. Fluids* **20**:1589 (1977).
7. M. H. Ernst, *Phys. Lett.* **69A**:390 (1979).
8. H. Cornille and A. Gervois, *Phys. Lett.* **79A**:291 (1980).
9. E. M. Hendriks and M. H. Ernst, *Physica* **120A**:545 (1983).
10. C. Cercignani, *Theory and Application of the Boltzmann Equation* (Scottish Academic Press, Edinburgh, 1975).
11. S. Simons, *Phys. Lett.* **69A**:239 (1978).
12. R. O. Barrachina, D. H. Fujii and C. R. Garibotti, *Phys. Lett.* **109A**:447 (1985).
13. R. Courant and D. Hilbert, *Methods of Mathematical Physics*, Vol. I (Interscience, New York, 1953).
14. P. Henrici, *Applied and Computational Complex Analysis*, Vol. 2 (Wiley, New York, 1977).
15. M. Barnsley and H. Cornille, *J. Math. Phys.* **25**:1176 (1980).
16. E. M. Hendriks and T. M. Nieuwenhuizen, *J. Stat. Phys.* **29**:591 (1982).
17. A. V. Bobylev, *Sov. Phys. Dokl.* **25**:257 (1980).
18. D. H. Fujii, R. O. Barrachina and C. R. Garibotti, *J. Stat. Phys.* **44**:95 (1986).
19. G. A. Baker, Jr., *Essential of Padé Approximants* (Academic Press, New York, 1975).
20. R. G. Gordon, *J. Math. Phys.* **9**:655 (1968).
21. H. Rutishauser, *Der Quotienten-Differenzen Algorithmus* (Birkhauser, Basel/Stuttgart, 1957).
22. M. Barnsley and G. Turchetti, *Lett. Nuovo Cimento* **26**:188 (1979).